Polyhedral risk measures in stochastic programming

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GAMS Workshop, Heidelberg, Sept. 1-3, 2003

DFG research center Berlin
mathematics for key technologies

1 Introduction

When wishing to replace the usual expectation-based objective by some functional measuring risk, at least the following three issues have to be addressed:

- What is an appropriate risk measure for the underlying practical model ?
- Does this exchange lead to serious changes in structure and stability ?
- Does the exchange cause serious computational problems ?

References: Schultz/Tiedemann 02, Schultz 03

Of course, the stochastic programming user wishes that his choice of a risk measure leads to the answer no on the last two questions.

We will discuss a subclass of convex risk measures having this enjoyable property.

Our motivation stems from stochastic programming applications in electricity portfolio management, i.e., from solving large scale mixed-integer stochastic programs.

2 Polyhedral risk measures

Let $\mathcal{Z}$ denote a linear space of real random variables on some probability space $(\Omega, \mathcal{F}, I\!P)$. We assume that $\mathcal{Z}$ contains the constants. A functional $\rho : \mathcal{Z} \to I\!R$ is called a risk measure if it satisfies the following two conditions for all z, $\tilde{z} \in \mathcal{Z}$:

(i) If $z \leq \tilde{z}$, then $\rho(z) \geq \rho(\tilde{z})$ (monotonicity).

(ii) For each $r \in I\!R$ we have $\rho(z + r) = \rho(z) - r$ (translation invariance).

A risk measure ρ is called convex if it satisfies the condition

$$\rho(\lambda z + (1 - \lambda)\tilde{z}) \leq \lambda \rho(z) + (1 - \lambda)\rho(\tilde{z})$$

for all z, $\tilde{z} \in \mathcal{Z}$ and $\lambda \in [0, 1]$.
A convex risk measure is called coherent if it is positively homogeneous, i.e., $\rho(\lambda z) = \lambda \rho(z)$ for all $\lambda \geq 0$ and $z \in \mathcal{Z}$.

References: Artzner/Delbaen/Eber/Heath 99, Föllmer/Schied 02

Definition: A risk measure ρ on $\mathcal{Z}$ will be called polyhedral if there exist $k, l \in \mathbb{N}$, $a, c \in \mathbb{R}^k$, $q, w \in \mathbb{R}^l$, a polyhedral set $X \subseteq \mathbb{R}^k$ and a polyhedral cone $Y \subseteq \mathbb{R}^l$ such that

$$\rho(z) = \inf \{\langle c, x\rangle + \mathbb{E}[\langle q, y\rangle] : \langle a, x\rangle + \langle w, y\rangle = z, x \in X, y \in Y\}$$

for each $z \in \mathcal{Z}$. Here, $\mathbb{E}$ denotes the expectation on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\langle \cdot, \cdot \rangle$ the scalar product on $\mathbb{R}^k$.

The notion *polyhedral* risk measure is motivated by the polyhedrality of $\rho(z)$ as a function of the scenarios of z if z is discrete.

Assume that ρ is a polyhedral risk measure on the space $\mathcal{Z} = L_1(\Omega, \mathcal{F}, \mathbb{P})$, that $\langle w, Y\rangle = \mathbb{R}$ and $\{u \in \mathbb{R} : uw - q \in Y^*\} \neq \emptyset$, where Y^* is the polar cone of Y. Then there exist two real numbers u_ℓ, $\ell = 1, 2$, such that

$$\rho(z) = \inf_{x \in X} \{\langle c, x\rangle + \mathbb{E}\,[\max_{\ell=1,2} u_\ell(z - \langle a, x\rangle)]\}.$$

In particular, ρ is a *convex* risk measure. It is *coherent* if X is a cone.

Proposition: Let $k, l \in I\!N$, $a, c \in I\!R^k$, $q, w \in I\!R^l$, a polyhedral set $X \subseteq I\!R^k$ and a polyhedral cone $Y \subseteq I\!R^l$ be given such that

$$\rho(z) = \inf\{\langle c, x\rangle + I\!E[\langle q, y\rangle] : \langle a, x\rangle + \langle w, y\rangle = z,\ x \in X, y \in Y\}$$

for each $z \in L_1(\Omega, \mathcal{F}, I\!P)$.
Let $\langle w, Y\rangle = I\!R$ and $\emptyset \neq \{u \in I\!R : uw - q \in Y^*\} \subset I\!R_-$ and a, c and X have the form $a = (\hat{a}, -1)$, $c = (\hat{c}, 1)$ and $X = \hat{X} \times I\!R$, where $\hat{a}, \hat{c} \in I\!R^{k-1}$ and $\hat{X} \subseteq I\!R^{k-1}$.
Then ρ is a convex risk measure on $L_1(\Omega, \mathcal{F}, I\!P)$ if it is finite.
Furthermore, if ρ is finite it admits the following dual representation

$$\begin{aligned}\rho(z) \;=\; \sup\{ -I\!E[\lambda z] + \inf_{\hat{x} \in \hat{X}} \langle \hat{c} + \hat{a}, \hat{x}\rangle : \lambda \in L_{p'}(\Omega, \mathcal{F}, I\!P),\\ I\!E[\lambda] = 1,\ -(q + \lambda w) \in Y^*\}\end{aligned}$$

for each $z \in L_p(\Omega, \mathcal{F}, I\!P)$ with $1 < p < +\infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof: by relying on stochastic programming methodology; the Lagrangian dual function has the form $D(\lambda) = \inf_{x \in X}\{\langle c + I\!E[\lambda]a, x\rangle + \inf_{y \in L_p, y \in Y} I\!E[\langle q + \lambda w, y\rangle] - I\!E[\lambda z]\}$ (Reference: Wets 70).

Example 1: Let $k = l = 1$, $a = -1$, $c = 1$ and $X = \mathbb{R}$. Then the conditions in the Proposition imply $Y = \mathbb{R}$, $w \neq 0$ and $\frac{q}{w} < 0$. We obtain that ρ has the form

$$\rho(z) = \mathbb{E}[\frac{q}{w}z] + \inf\{(1 + \frac{q}{w})x : x \in \mathbb{R}\}.$$

Hence, ρ is finite iff $\frac{q}{w} = -1$ iff $\rho(z) = \mathbb{E}[-z]$.

Example 2: Conditional, Tail or Average Value at Risk
We consider the *Average Value at Risk* $AVaR_\alpha$ defined by

$$\begin{aligned} AVaR_\alpha(z) &:= \frac{1}{\alpha}\int_0^\alpha VaR_\gamma(z)d\gamma \\ &= \inf_{r \in \mathbb{R}} \{r + \frac{1}{\alpha}\mathbb{E}[\max\{0, -r - z\}]\}, \end{aligned}$$

where $VaR_\alpha(z) := \inf\{r \in \mathbb{R} : \mathbb{P}(z + r < 0) \leq \alpha\}$ is the Value at Risk at level $\alpha \in (0, 1)$.
$AVaR_\alpha$ is polyhedral by setting $k = 1$, $l = 2$, $a = -1$, $c = 1$, $q = (\frac{1}{\alpha}, 0)$, $w = (-1, 1)$, $X = \mathbb{R}$ and $Y = \mathbb{R}^2_+$.
The condition $-(q + \lambda w) \in Y^*$ in the dual representation is equivalent to $\lambda \in [0, \frac{1}{\alpha}]$.

3 Multiperiod polyhedral risk measures

While the notion of a polyhedral risk measure is appropriate for two-stage stochastic programming models, we now consider a multiperiod extension in case that instead of the real random variable z a real stochastic process $z = \{z_t, \mathcal{F}_t\}_{t=2}^T$ with a filtration $\mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}_t \subseteq \cdots \subseteq \mathcal{F}_T = \mathcal{F}$ is given. It is assumed that z_τ, $\tau = 2, \ldots, t$, is measurable with respect to $\mathcal{F}_t$, $t = 2, \ldots, T$.

As natural candidates for such an extension we consider functionals ρ on the linear space $\times_{t=2}^T L_1(\Omega, \mathcal{F}_t, \mathbb{P})$ that are defined as optimal values of specific multi-stage stochastic programs. Namely, we assume that there are $k_t \in \mathbb{N}$, $c_t \in \mathbb{R}^{k_t}$, $t = 1, \ldots, T$, $a_t \in \mathbb{R}^{k_{t-1}}$, $w_t \in \mathbb{R}^{k_t}$, $t = 2, \ldots, T$, a polyhedral set $Y_1 \subseteq \mathbb{R}^{k_1}$ and polyhedral cones $Y_t \subseteq \mathbb{R}^{k_t}$, $t = 2, \ldots, T$, such that

$$\rho(z) = \inf\{\mathbb{E}[\sum_{t=1}^{T} \langle c_t, y_t \rangle] \ : \ y_1 \in Y_1, y_t \text{ is } \mathcal{F}_t\text{-measurable}, y_t \in Y_t,$$
$$\langle a_t, y_{t-1} \rangle + \langle w_t, y_t \rangle = z_t, t = 2, \ldots, T\}$$

Dynamic programming formulation of ρ:

$$\begin{aligned}
\rho(z) &= \inf\{\langle c_1, y_1\rangle + I\!\!E[v_2(y_1, z)] : y_1 \in Y_1\} \\
v_t(y_{t-1}, z_t, \ldots, z_T) &:= \inf\{\langle c_t, y_t\rangle + I\!\!E[v_{t+1}(y_t, z_{t+1}, \ldots, z_T)|\mathcal{F}_t] : \\
&\qquad y_t \in Y_t, \langle a_t, y_{t-1}\rangle + \langle w_t, y_t\rangle = z_t\}, \\
&\qquad t = T, \ldots, 2, \\
v_{T+1}(y_T) &:= 0\,.
\end{aligned}$$

References: Rockafellar/Wets 76, Evstigneev 76

Dual formulation for ρ:

$$\begin{aligned}
\rho(z) = \sup\{&\inf_{y_1 \in Y_1} \langle c_1 + I\!\!E[\lambda_2]a_2, y_1\rangle - I\!\!E\left[\sum_{t=2}^{T} \lambda_t z_t\right] : \\
&\lambda_t \in L_{p'}(\Omega, \mathcal{F}_t, I\!\!P), t = 2, \ldots, T, -(c_T + \lambda_T w_T) \in Y_T^*, \\
&-(c_t + w_t\lambda_t + a_{t+1}I\!\!E[\lambda_{t+1}|\mathcal{F}_t]) \in Y_t^*, t = T-1, \ldots, 2\}
\end{aligned}$$

holds whenever $z_t \in L_p(\Omega, \mathcal{F}_t, I\!\!P)$, $t = 2, \ldots, T$, $p > 1$, $\frac{1}{p} + \frac{1}{p'} = 1$, and the right-hand side is finite.

References: Eisner/Olsen 75, Rockafellar/Wets 76, 78

Definition: (Artzner/Delbaen/Eber/Heath/Ku 02)
A functional ρ on (a subset of) $\times_{t=2}^{T} L_p(\Omega, \mathcal{F}_t, I\!P)$ $(p > 1)$ is called a multiperiod coherent risk measure if it satisfies the Fatou property and if there exist positive reals $\eta_t > 0$, $t = 2, \ldots, T$, $\sum_{t=2}^{T} \eta_t = 1$, and a closed convex set Λ of nonnegative functions contained in $\times_{t=2}^{T} L_{p'}(\Omega, \mathcal{F}_t, I\!P)$, with $\frac{1}{p} + \frac{1}{p'} = 1$, such that

$$\rho(z) = \sup\{-\sum_{t=2}^{T} \eta_t I\!E[\lambda_t z_t] : \lambda \in \Lambda, \sum_{t=2}^{T} \eta_t I\!E[\lambda_t] = 1\}.$$

$$\begin{aligned} \Lambda(\eta) \;:=\; & \{\lambda \in \times_{t=2}^{T} L_{p'}(\Omega, \mathcal{F}_t, I\!P) : -(c_T + \lambda_T \frac{w_T}{\eta_T}) \in Y_T^*, \\ & -(c_t + \frac{w_t}{\eta_t}\lambda_t + \frac{a_{t+1}}{\eta_{t+1}} I\!E[\lambda_{t+1}|\mathcal{F}_t]) \in Y_t^*, t = 2, \ldots, T-1\} \end{aligned}$$

Proposition:
A multiperiod polyhedral risk measure is multiperiod coherent if
(i) $\eta_t = \frac{1}{T-1}$, $t = 2, \ldots, T$,
(ii) $\lambda \in \Lambda(\eta)$ implies $\lambda_t \geq 0$, $I\!P$-a.s., $t = 2, \ldots, T$, and
(iii) $\lambda \in \Lambda(\eta)$ and $\inf\limits_{y_1 \in Y_1} \langle c_1 + I\!E[\lambda_2] a_2, y_1 \rangle = 0$ imply $I\!E[\lambda_t] = 1$, $t = 2, \ldots, T$.

4 Multistage SP models with minimal risk

We consider a stochastic process $\{\xi^t, \mathcal{F}_t\}_{t=2}^T$ with distribution P modelling the uncertain future and the multi-stage stochastic program

$$\text{Minimize} \qquad \sum_{i=1}^{I} C_{i1} x_i^1 + \mathbb{E}\left[\sum_{t=2}^{T}\sum_{i=1}^{I} C_{it}(\xi^t) x_i^t\right] \qquad (\text{expectation})$$

or, alternatively,

$$\text{Minimize} \qquad \sum_{i=1}^{I} C_{i1} x_i^1 + \rho\left(\{\sum_{i=1}^{I} C_{it}(\xi^t) x_i^t\}_{t=2}^T\right) \qquad (\text{risk measure})$$

such that

$$\begin{gathered} x^t \text{ is } \mathcal{F}_t - \text{measurable }, x_i^t \in X_{it}, t = 1, \dots, T, \\ A_{it,t} x_i^t + A_{it,t-1}(\xi^t) x_i^{t-1} \geq g_{it}(\xi^t), t = 2, \dots, T, i = 1, \dots, I, \\ \textstyle\sum_{i=1}^{I} B_{it}(\xi^t) x_i^t \geq d_t(\xi^t), t = 1, \dots, T. \end{gathered}$$

Here, $x^t = (x_1^t, \dots, x_I^t)$, $t = 1, \dots, T$, $\mathcal{F}_1 = \{\emptyset, \Omega\}$, $\mathcal{F}_T = \mathcal{F}$, $A_{it,\tau}$, $\tau = t-1, t$, B_{it}, g_{it} and d_t are matrices and vectors possibly depending on ξ^t, $t = 1, \dots, T$, and X_{it} subsets of Euclidean spaces.

Incorporating the multiperiod risk measure into the original program leads to a multistage model exhibiting the same structure and having (x, y) as decision.

$$\min_{(x,y)} \sum_{i=1}^{I} C_{i1} x_i^1 + \mathbb{E}[\sum_{t=1}^{T} \langle c_t, y_t \rangle]$$

such that

$$y_1 \in Y_1, y_t \text{ is nonanticipative}, y_t \in Y_t,$$

$$\langle a_t, y_{t-1} \rangle + \langle w_t, y_t \rangle = \sum_{i=1}^{I} C_{it}(\xi^t) x_i^t, t = 2, \ldots, T,$$

$$x^t \text{ is nonanticipative }, x_i^t \in X_{it}, t = 2, \ldots, T,$$

$$A_{it,t} x_i^t + A_{it,t-1}(\xi^t) x_i^{t-1} \geq g_{it}(\xi^t), t = 2, \ldots, T, i = 1, \ldots, I,$$

$$\sum_{i=1}^{I} B_{it}(\xi^t) x_i^t \geq d_t(\xi^t), t = 1, \ldots, T.$$

The stability behaviour and the metrics μ_c as well as the decomposition structure (e.g. for scenario, node and geographic decomposition) do (almost) not change !