time occurs on line $l = 2$ in microperiods $s = 1$ and $s = 2$. Therefore, $x^{e}_{21} > 0$ and $x^{b}_{22} > 0$.

According to the introductory remarks above, we use the following notation for the model:

Indices	
i, j, v	$= 1, \ldots, J$ products, whereas =0 means neutral
k, l	$= 1, \ldots, L$ production lines (multi-stage and/or parallel)
t	$= 1, \ldots, T$ macroperiods (e.g., weeks, months)
	$= T + 1$ artificial macroperiod
s	$= 1, \ldots, S$ microperiods
	$= S + 1$ artificial microperiod
Index sets	
$\mathcal{N}_j$	Set of all direct and indirect successors of product j
$\mathcal{N}^I_j \subseteq \mathcal{N}_j$	Set of all immediate successors of product j
$\mathcal{I}_l$	Set of products that can be produced on production line l
$\mathcal{D}$	Set of all (k, i, l, j)-tuples consisting of line-product combinations (k, i) and (l, j) where product j is a direct successor of i ($j \in \mathcal{N}^I_i$) and j is producible on line l ($a_{lj} > 0$) and i on k ($a_{ki} > 0$)
Φ	Set of all microperiods with fixed starting times
Λ	Set of all last microperiods of macroperiods
Π_l	Set of all microperiods in which production on line l is not allowed
Data	
a_{lj}	Capacity consumption (time) needed to produce one unit of product j on line l
m_{lj}	Minimum lot-size of product j (units) if produced on line l
m_{l0}	Minimum time line l has to remain shut down
h_{js}	Holding costs of product j (per unit and per macroperiod t with $s = f_{t+1} - 1$)
c_{lj}	Production costs of product j (per unit) on line l
b_l	Standby costs on line l
y_{lj0}	Equals 1, if line l is set up for product j at the beginning of planning (0 otherwise)
s_{lij}	Setup costs of a changeover from product i to product j on line l
st_{lij}	Setup time of a changeover from product i to product j on production line l
d_{js}	Demand for product j in microperiod s (units)
I_{j0}	Initial inventory of product j at the beginning of planning (units)
$\overline{w}_s$	Starting time of fixed period $s \in \Phi$
p_{ji}	Number of units of product j required to produce one unit of the direct or indirect successor i
$I^{\max}$	Maximum stock level (units)
$W^{\max}_{lj}$	Maximum WIP-stock level after production on line l (units)

e_j	Purchasing costs (per unit) of product j
$e_j^{\max}$	Maximum number of units of product j that can be externally purchased
g	Overtime costs
$g^{\max}$	Maximum overtime
Variables	
$w_s \geq 0$	Starting time of microperiod s
$I_{js} \geq 0$	Inventory of product j at the end of microperiod s (units)
$x^{\mathrm{b}}_{ls} \geq 0$	Fractional setup time for changeover at the beginning of period s on line l
$x^{\mathrm{e}}_{ls} \geq 0$	Fractional setup time for changeover at the end of period s on line l
$x_{ljs} \geq 0$	Total quantity of product j produced in microperiod s on line l (units)
$\widehat{x}_{ljs} \geq 0$	Share of x_{ljs} that can be used by successors in the same microperiod s (units)
$\vec{x}_{ljs} \geq 0$	Share of x_{ljs} that can as WIP-stock first be used by successors in the following microperiod $s+1$ (units)
$\overline{x}^{\mathrm{b}}_{ls} \geq 0$	Standby time on line l in microperiod s before production
$\overline{x}^{\mathrm{e}}_{ls} \geq 0$	Standby time on line l in microperiod s after production
$o_{js} \geq 0$	Externally purchased quantity of product j in microperiod s (units)
$r_s \geq 0$	Overtime used in microperiod s
$y_{ljs} \in \{0, 1\}$	Setup state: $y_{ljs} = 1$, if line l is set up for product j in microperiod s (0 otherwise)
$z_{lijs} \geq 0$	Takes on 1, if a changeover from product i to product j takes place on line l during microperiod s (0 otherwise)

Objective function

$$\min \sum_{s\in\Lambda, j\neq 0} h_{js} I_{js} + \sum_{l,i,j,s} s_{lij} z_{lijs} + \sum_{l,j,s} c_{lj} x_{ljs}$$

$$+ \sum_{l,s} b_l(\overline{x}^{\mathrm{b}}_{ls} + \overline{x}^{\mathrm{e}}_{ls}) + \sum_{j\neq 0,s} e_j o_{js} + \sum_s g \cdot r_s + \sum_{l,s\in\Lambda, j\neq 0} h_{js}\vec{x}_{ljs} \tag{1}$$

subject to

$$w_s = \overline{w}_s \quad \forall s \in \Phi \tag{2}$$

$$\widehat{x}_{ljs} + \vec{x}_{ljs} = x_{ljs} \quad \forall l, j \neq 0, s \tag{3}$$

$$I_{js} = I_{j,s-1} + \sum_l \widehat{x}_{ljs} + \sum_l \vec{x}_{lj,s-1} + o_{js} - d_{js} - \sum_l \sum_{i\in N_j^I} p_{ji} x_{lis} \forall j \neq 0, s \tag{4}$$

$$I_{js} \le I^{\max} \quad \forall j \ne 0, s \tag{5}$$
$$\vec{x}_{ljs} \le \mathrm{W}_{lj}^{\max} \quad \forall l, j, s \tag{6}$$
$$I_{jS} = I_{j0} \quad \forall j \ne 0 \tag{7}$$
$$x_{l1}^{\mathrm{b}} = \sum_{i,j} st_{lij} z_{lij1} \quad \forall l \tag{8}$$
$$x_{l,s-1}^{\mathrm{e}} + x_{ls}^{\mathrm{b}} = \sum_{i,j} st_{lij} z_{lijs} \quad \forall l, s \ge 2 \tag{9}$$
$$x_{ls}^{\mathrm{b}} + \overline{x}_{ls}^{\mathrm{b}} + \sum_{j} a_{lj} x_{ljs} + \overline{x}_{ls}^{\mathrm{e}} + x_{ls}^{\mathrm{e}} = (w_{s+1} - w_s) + r_s \quad \forall l, s \tag{10}$$
$$a_{lj} x_{ljs} \le \overline{w}_{S+1} y_{ljs} \quad \forall l, j, s \tag{11}$$
$$x_{ljs} \ge m_{lj}(y_{ljs} - y_{lj,s-1}) \quad \forall l, j, s \tag{12}$$
$$\sum_{j} y_{ljs} = 1 \quad \forall l, s \tag{13}$$
$$y_{ljs} = 0 \quad \forall l, j \notin \mathcal{I}_l, s \tag{14}$$
$$y_{li,s-1} + y_{ljs} - 1 \le z_{lijs} \quad \forall l, i, j, s \tag{15}$$
$$\sum_{i,j} z_{lijs} = 1 \quad \forall l, s \tag{16}$$
$$\sum_{j \ne 0} x_{ljs} = 0 \quad \forall l, s \in \Pi_l \tag{17}$$
$$o_{js} \le e_j^{\max} \quad \forall j \ne 0, s \tag{18}$$
$$r_s \le g^{\max} \quad \forall s \in \Lambda \tag{19}$$
$$r_s = 0 \quad \forall s \notin \Lambda \tag{20}$$
$$x_{ls}^{\mathrm{b}} + \overline{x}_{ls}^{\mathrm{b}} \ge x_{ks}^{\mathrm{b}} + \overline{x}_{ks}^{\mathrm{b}} - \overline{w}_{S+1}(2 - y_{ljs} - y_{kis}) \quad \forall s, (k, i, l, j) \in \mathcal{D} \tag{21}$$
$$a_{ki} \widehat{x}_{kis} + \overline{x}_{ks}^{\mathrm{e}} + x_{ks}^{\mathrm{e}} \ge \overline{x}_{ls}^{\mathrm{e}} + x_{ls}^{\mathrm{e}} - \overline{w}_{S+1}(2 - y_{kis} - y_{ljs}) \forall s, (k, i, l, j) \in \mathcal{D} \tag{22}$$

The objective is to minimize the sum of holding costs of the lot-sizing stock, sequence-dependent setup costs, and production costs, as well as costs for standby, external purchase, overtime, and holding of WIP-stock (1). With the help of (2), the starting times of all microperiods in Φ (including the macroperiods) are fixed. The quantity split is allowed by (3), which divides the production quantity x_{ljs} into a part $\widehat{x}_{ljs}$ that is directly available in the same period s and into $\vec{x}_{ljs}$ that is first available in the next microperiod $s + 1$.

The inventory balancing constraints (4) ensure that primary as well as secondary demand is met without backlogging. More precisely, the inventory of a certain product j at the end of microperiod s equals the inventory of the same product at the end of the preceding microperiod plus the total inflow on stock minus the total outflow from stock during period s. The inflow on stock is composed by the total production of j during s, the WIP-stock of j that has been built up in $s - 1$, and the externally purchased quantities. Note that this WIP-stock has a lead time of a single, usually quite short microperiod and thus allows a more realistic modeling than commonly used BTB-models, as already mentioned in Sects. 1 and 2. The outflow is the primary demand of j in s and the secondary demand generated by direct successors i that are also produced in s. Constraints (5) restrict the overall inventory of each product if necessary, whereas the line-specific WIP-stock can be limited by (6). Furthermore, constraints (7) prevent zero "end-of-horizon" inventories, which may have a negative impact on later periods beyond the planning horizon (Stadtler 2000). Note that a maximum stock level for each product is necessary, if for instance the goods are perishable. A further realistic assumption might be to limit the cumulative inventory for all products. This can be easily achieved by constraints built analogously. The same is true for the WIP stock.

Constraints (8) and (9) ensure that the required setup time st_{lij} of a changeover from i to j is actually used and thus make the setup split possible. These two groups of constraints are improved as compared to Meyr (2004). First, a changeover at the beginning of microperiod $s = 1$ is now allowed. Second, the setup time fractions $x^{\mathrm{b}}_{ls}, x^{\mathrm{e}}_{ls}$ are only considered in an aggregate manner (sum over i, j) compared to the original formulation, thus reducing the number of variables. This is possible without a loss of details, since the synchronization constraints have also changed as described later on. According to the time structure outlined above, constraints (10) build up a microperiod s, which consists of setup time fractions, idle and production time. The length of this microperiod equals the time interval $w_{s+1} - w_s$ between the two subsequent microperiods s and $s + 1$, which can be prolonged if overtime r_s is used. Since the starting times of some microperiods are fixed by (2), all-in-all limited production capacity can be respected.

Because of (11) production can only take place if the line is set up accordingly. For this "setup forcing constraint", the right-hand side needs a coefficient of sufficient size. This coefficient, sometimes also denoted as "big M", has to be chosen carefully as it should not limit the production unnecessarily, but also should not be unnecessarily big. Thus the fixed end of the planning horizon $\overline{w}_{S+1}$ has been proposed.

Constraints (12) enforce a minimum lot-size and are needed because setup costs (or times) do not always satisfy the triangular inequality $s_{liv} + s_{lvj} \geq s_{lij}$.

Equation (13) enforce that a line can only be set up for exactly one product per microperiod. Since not all products can be produced on every line, (14) forbid irrelevant combinations. (15) link the setup state indicators y with the changeover indicators z. Together with the objective function (1) they ensure that z_{lijs} is only set to 1 if line l was set up for i in $s - 1$ and for j in s. Correspondingly, z_{lijs} can be defined as continuous variables. Note that since a setup for the same product j in two consecutive microperiods $s - 1$ and s is possible and since $s_{ljj} = st_{ljj} = 0$ holds for all l and j, the corresponding variables z_{ljjs} directly allow a setup carryover. That means, in contrast to other formulations like the Capacitated Lot-Sizing Problem with Linked lot sizes (CLSPL) (Haase 1994; Suerie and Stadtler 2003), no additional variables for a setup carryover are necessary. Moreover, when an idle period s without any production activities occurs on a line l ($\sum_{j>0} x_{ljs} = 0$), it is up to the model to decide whether

- the setup state should be conserved for the last product j produced on this line ($y_{lj,s-1} = 1$) without incurring any setup costs or times ($z_{ljjs} = z_{ljj,s+1} = 1$),

- a changeover to another product $i > 0, i \neq j$ should be executed ($z_{ljjs} = z_{lji,s+1} = 1$), or
- the setup state should get lost (by changing to the fictitious product $j = 0$) because the production line should be shut down after period $s-1$ and started up again at the beginning of period $s+1$ ($z_{lj0s} = z_{l0i,s+1} = 1$).

Note that most models for simultaneous lot-sizing and scheduling only allow modeling either a conservation or a loss of the setup state after idle periods, but not both. To get a tighter formulation, (16) are added (but not necessarily needed).

With respect to e.g., working calendars or pre-determined maintenance activities (17) allow production to be prohibited on certain lines in certain microperiods. Constraints (18), (19), and (20) impose limits on external purchasing and overtime, respectively. Note that only the capacity of fixed microperiods can be extended by overtime.

Finally, the multi-stage production enforces the additional synchronization constraints (21) and (22) to guarantee feasible plans. These constraints are new and make the major difference to Meyr (2004). When considering a predecessor line k producing a predecessor product i and a successor line l producing a (direct) successor product j, both equations ensure that the production of j must neither start before production of product i starts nor end before the (relevant) production of product i (which is needed in the same period) ends. Of course, a necessary prerequisite is that both products can be produced on the respective lines at all, i.e., that $a_{ki} > 0$ and $a_{lj} > 0$. This is expressed by the index set $\mathcal{D}$, which contains all tuples (k, i, l, j) that fulfill these conditions.

As illustrated in Fig. 4 constraints (21) force setup and idle time before production on successor line l to be at least as long as setup and idle time before production on line k. Constraints (22) assure that setup and idle time after production on line l are at most as long as the time needed on line k for producing the parts which are first available in the next microperiod, standby and setup at the end of the current microperiod. However, these conditions only make sense if pre-product i is actually produced on k and successor j on l. Accordingly, both types of constraints are only "active", if products i and j are set up on lines k and l, respectively. Since (21) and (22) prevent an incorrect timing *for each* predecessor-successor relation and since the inventory balancing constraints (4) ensure *aggregate* material availability, the resulting production schedules appear fairly realistic even for parallel (predecessor and/or successor) lines.

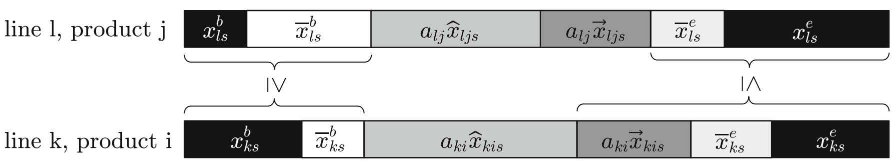

Fig. 4 Production of successor product j on line l must neither start nor end before predecessor product i on line k in every microperiod s.