Equilibrium problem of a two player game:

$$(v_1^*, v_2^*) \in \arg\max_{(v_1,v_2)\in\mathcal{V}_d} \left(\mathbb{E}_{\eta^M}\left[-v_1^T\xi_d(\theta_1)\theta_1 + \frac{v_1^T\xi_d(\theta_1)}{v_1^T\xi_d(\theta_1) + (v_2^*)^T\xi_d(\theta_2)} \right] + \mathbb{E}_{\eta^M}\left[-v_2^T\xi_d(\theta_2)\theta_2 + \frac{v_2^T\xi_d(\theta_2)}{{v_1^*}^T\xi_d(\theta_1) + (v_2)^T\xi_d(\theta_2)} \right] \right),$$

where

$$(\mathcal{V}_d) := \left\{ (v_1, v_2) \in \mathbb{R}^d \times \mathbb{R}^d : 0 \le v_1^T\xi_d(\theta_1^i) \le 1/\alpha_1, 0 \le v_2^T\xi_d(\theta_2^i) \le 1/\alpha_2 \; \forall i = 1, \cdots, M \right\},$$

and $\theta_1^i, \theta_2^i, i = 1, \cdots, M$ are samples, and $\xi_d(\theta) = [1, \theta, \theta^2, \cdots, \theta^{d-1}]^T \in \mathbb{R}^d$, η^M is the uniform distribution over the sample points (θ_1^i, θ_2^j) for $i, j = 1, \cdots, M$.

Lagrange function:

$$\begin{aligned}
& L(v_1, v_2, \lambda_1, \lambda_2, \mu_1, \mu_2) \\
= \; & -\left(\mathbb{E}_{\eta^M}\left[-v_1^T\xi_d(\theta_1)\theta_1 + \frac{v_1^T\xi_d(\theta_1)}{v_1^T\xi_d(\theta_1) + (v_2^*)^T\xi_d(\theta_2)} \right] + \mathbb{E}_{\eta^M}\left[-v_2^T\xi_d(\theta_2)\theta_2 + \frac{v_2^T\xi_d(\theta_2)}{{v_1^*}^T\xi_d(\theta_1) + (v_2)^T\xi_d(\theta_2)} \right] \right) \\
& -\sum_{i=1}^M \mu_1^i(v_1^T\xi_d(\theta_1^i)) - \sum_{i=1}^M \mu_2^i(v_2^T\xi_d(\theta_2^i)) + \sum_{i=1}^M \lambda_1^i(v_1^T\xi_d(\theta_1^i) - 1/\alpha_1) + \sum_{i=1}^M \lambda_2^i(v_2^T\xi_d(\theta_2^i) - 1/\alpha_2)
\end{aligned}$$

Optimality conditions:

$$\begin{aligned}
& \mathbb{E}_{\eta^M}\left[\xi_d(\theta_1)\theta_1 - \frac{{v_2^*}^T\xi_d(\theta_2)\xi_d(\theta_1)}{({v_1^*}^T\xi_d(\theta_1) + (v_2^*)^T\xi_d(\theta_2))^2} \right] - \sum_{i=1}^M \mu_1^i\xi_d(\theta_1^i) + \sum_{i=1}^M \lambda_1^i\xi_d(\theta_1^i) = 0, \\
& \mathbb{E}_{\eta^M}\left[\xi_d(\theta_2)\theta_2 - \frac{{v_1^*}^T\xi_d(\theta_1)\xi_d(\theta_2)}{({v_1^*}^T\xi_d(\theta_1) + (v_2^*)^T\xi_d(\theta_2))^2} \right] - \sum_{i=1}^M \mu_2^i\xi_d(\theta_2^i) + \sum_{i=1}^M \lambda_2^i\xi_d(\theta_2^i) = 0, \\
& 0 \le \mu_1^i \perp {v_1^*}^T\xi_d(\theta_1^i) \ge 0, i = 1, \cdots, M, \\
& 0 \le \mu_2^i \perp {v_2^*}^T\xi_d(\theta_2^i) \ge 0, i = 1, \cdots, M, \\
& 0 \le \lambda_1^i \perp -{v_1^*}^T\xi_d(\theta_1^i) + 1/\alpha_1 \ge 0, i = 1, \cdots, M, \\
& 0 \le \lambda_2^i \perp -{v_2^*}^T\xi_d(\theta_1^i) + 1/\alpha_2 \ge 0, i = 1, \cdots, M,
\end{aligned}$$

which is a nonlinear complemetarity problem. We use PATH to solve the NCP.

In the actually implementation of the code, we

$$\begin{aligned}
0 \le \mu_1^i \perp {v_1^*}^T\xi_d(\theta_1^i) \ge 0, i = 1, \cdots, M, \\
0 \le \mu_2^i \perp {v_2^*}^T\xi_d(\theta_2^i) \ge 0, i = 1, \cdots, M,
\end{aligned}$$

by

$$\begin{aligned}
0 \le \mu_1^i \perp {v_1^*}^T\xi_d(\theta_1^i) - \epsilon \ge 0, i = 1, \cdots, M, \\
0 \le \mu_2^i \perp {v_2^*}^T\xi_d(\theta_2^i) - \epsilon \ge 0, i = 1, \cdots, M,
\end{aligned}$$

where ϵ is a small positive number such as 0.001. This is to ensure the denominators in the first two equations to be positive.

The issues that we encountered are:

1. If $\theta_1 \sim Uniform(0.01, 1.01)$ and $\theta_2 \sim Uniform(0.51, 5.51)$, the solution can't be found.
2. We are not sure whether the code (see attached) is correct or not.
3. When we change ϵ value, the solution changes drastically.